2008 BLUE MOP, INEQUALITIES-II

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(1) (Tournament of Towns-97) Let a, b, c be positive numbers such that $abc = 1$. Prove that
$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \le 1.$$

(2) (IMO-95) Let $a, b, c > 0$ such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

(3) Let x, y, z be non-negative real numbers with $xy + yz + zx = 1$. Prove that
$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \ge \frac{5}{2}.$$

(4) If $a, b, c > 0$, prove that
$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \ge \frac{3(ab + bc + ca)}{a + b + c}.$$

(5) If $x + y + z = 1$ for some non-negative numbers x, y, z, prove that
$$0 \le xy + yz + zx - 2xyz \le \frac{7}{27}.$$

(6) (Iran-96) Let x, y, z be positive numbers. Prove that
$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \ge \frac{9}{4}.$$

(7) (IMO-05) Let x, y and z be positive numbers such that $xyz \ge 1$. Prove that
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \ge 0.$$

(8) (IMO-00) Let a, b, c be positive numbers such that $abc = 1$. Prove that
$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

Problem 1, Solution by Ali Gurel: We multiply out and clear the denominators. After using the relation $abc = 1$ to homogenize both sides, we get

$$\begin{aligned} LHS &= \frac{1}{2}\left[\frac{7}{3},\frac{1}{3},\frac{1}{3}\right] + 2\left[\frac{5}{3},\frac{2}{3},\frac{2}{3}\right] + \frac{3}{2}\left[\frac{4}{3},\frac{4}{3},\frac{1}{3}\right] + \frac{1}{2}[1,1,1] \\ RHS &= \frac{1}{2}\left[\frac{7}{3},\frac{1}{3},\frac{1}{3}\right] + \left[\frac{5}{3},\frac{2}{3},\frac{2}{3}\right] + \frac{3}{2}\left[\frac{4}{3},\frac{4}{3},\frac{1}{3}\right] + [2,1,0] + \frac{1}{2}[1,1,1] \end{aligned}$$

Finally $LHS \geq RHS$ since by Muirhead $\left[\frac{5}{3},\frac{2}{3},\frac{2}{3}\right] \geq [2,1,0]$ $\square$

Problem 2, Solution by Zhifan Zhang: We will show that

$$\sum_{cyc} b^3c^3(a+b)(c+a) \geq \frac{3}{2}a^3b^3c^3(a+b)(b+c)(c+a).$$

Expanding, we get

$$\begin{aligned} LHS &= [4,3,1] + \frac{1}{2}[3,3,2] + \frac{1}{2}[4,4,0] \\ &= \left[\frac{16}{3},\frac{13}{3},\frac{7}{3}\right] + \frac{1}{2}\left[\frac{13}{3},\frac{13}{3},\frac{10}{3}\right] + \frac{1}{2}\left[\frac{16}{3},\frac{16}{3},\frac{4}{3}\right] \\ RHS &= \frac{3}{2}[5,4,3] + \frac{1}{2}[4,4,4] \end{aligned}$$

By Muirhead

$$\begin{aligned} &\left[\frac{16}{3},\frac{13}{3},\frac{7}{3}\right] \geq [5,4,3], \\ &\left[\frac{16}{3},\frac{16}{3},\frac{4}{3}\right] \geq [5,4,3], \quad \text{and} \\ &\left[\frac{13}{3},\frac{13}{3},\frac{10}{3}\right] \geq [4,4,4]. \end{aligned}$$

combining these three inequalities, we get $LHS \geq RHS$ $\square$

Problem 3, Solution by Gye Hyun Baek: Using the relation $xy + yz + zx = 1$ we get the equivalent inequality:

$$11 + 4(x^4 + y^4 + z^4) \geq 25x^2y^2 + z^2 + (x^2 + y^2 + z^2) + 17(x^2y^2 + y^2z^2 + z^2x^2).$$

Now we make every term to degree six by multiplying with a power of $xy+yz+zx = 1$. After the cancelations, we get

$$LHS - RHS = \left(4\sum x^5y + 3\sum x^4yz + 14\sum x^3y^2z + 41x^2y^2z^2\right) - \left(\sum x^4y^2 + 3\sum x^3y^3\right).$$

We notice that, by Muirhead

$$4\sum x^5y \geq \sum x^4y^2 + 3\sum x^3y^3.$$

Thus, $LHS \geq RHS$, as desired. The equality holds when $(x, y, z) = (0, 1, 1), (1, 0, 1)$, or $(1, 1, 0)$ $\square$

Problem 4, Solution by Toan Phan: Firstly, we will prove that

$$\sum_{cyc} a \geq \frac{3\sum_{cyc} bc}{\sum_{cyc} a}$$

Multiplying out we see that this is equivalent to

$$[2,0,0] \geq [1,1,0]$$

which follows by Muirhead. Secondly, we will prove that

$$\sum_{cyc} \frac{a^3}{b^2 - bc + c^2} \geq \sum_{cyc} a.$$

And this one becomes equivalent to

$$[6,4,0] + [6,3,1] + [4,3,3] + [9,1,0] \geq [4,3,3] + [7,3,0] + [6,3,1] + [6,4,0],$$

which is true since by Muirhead $[9,1,0] \geq [7,3,0]$. Finally, we are done by combining the two inequalities we proved $\square$

Problem 5, Solution by Damien Jiang: Let $S = xy + yz + zx - 2xyz$. Note that

$$S = \sum_{cyc} xy \sum_{cyc} x - 2xyz = \sum_{sym} x^2y + xyz.$$

Clearly $S \geq 0$ because all terms in S are non-negative. Next, we show that $S \leq \frac{7}{27}(x+y+z)^3$. Expanding and canceling the likewise terms, we get the equivalent expression

$$6\sum_{sym} x^2y \leq 7\sum_{cyc} x^3 + 15xyz.$$

But this follows from Schur:

$$5\sum_{sym} x^2y \leq 5\sum_{cyc} x^3 + 15xyz$$

and Muirhead:

$$\sum_{sym} x^2y \leq 2\sum_{cyc} x^3 \ \square$$

Problem 6, Solution by Nicholas Triantafillou: After multiplying through and clearing the denominators, we get

$$\begin{aligned} LHS &= 4[5,1,0] + 10[4,1,1] + 8[4,2,0] + 6[3,3,0] + 52[3,2,1] + 16[2,2,2] \\ RHS &= 9[4,2,0] + 9[4,1,1] + 54[3,2,1] + 9[3,3,0] + 15[2,2,2] \end{aligned}$$

and

$$LHS - RHS = (4[5,1,0] - [4,2,0] - 3[3,3,0]) + xyz([3,0,0] + [1,1,1] - 2[2,1,0]) \geq 0,$$

where the last inequality follows from Murihead and Schur $\square$

Problem 7, Solution by Matthew Superdock: After expanding and canceling likewise terms we get

$$\sum_{sym}(x^5-x^2)(y^5+z^2+x^2)(z^5+x^2+y^2)\geq 0$$

$$\Leftrightarrow \quad \sum_{sym}(x^5y^5z^5+4x^7y^5+x^9+x^5y^2z^2)\geq \sum_{sym}(x^5y^5z^2+2x^5y^4+x^6+2x^4y^2+x^2y^2z^2)$$

By Muirhead and that $xyz\geq 1$, we have the following inequalities:

$$\begin{aligned}
\sum_{sym}x^5y^5z^5 &\geq \sum_{sym}x^2y^2z^2\\
\sum_{sym}x^7y^5 &\geq \sum_{sym}x^5y^5z^2\\
2\sum_{sym}x^7y^5 &\geq 2\sum_{sym}x^6y^5z\geq 2\sum_{sym}x^5y^4\\
\sum_{sym}x^9 &\geq \sum_{sym}x^7yz\geq \sum_{sym}x^6\\
\sum_{sym}x^7y^5 &\geq \sum_{sym}x^{\frac{13}{3}}y^{\frac{7}{3}}z^{\frac{1}{3}}\geq \sum_{sym}x^4y^2\\
\sum_{sym}x^7y^5 &\geq \sum_{sym}x^6y^4z^2\geq \sum_{sym}x^4y^2.
\end{aligned}$$

Combining all these, we are done $\square$

Problem 8, Solution by Minseon Shin: Since $abc=1$, there exists positive reals x,y,z such that $a=x/y$, $b=y/z$, $c=z/x$. Substituting into the inequality, we get:

$$\frac{(-x+y+z)(x-y+z)(x+y-z)}{xyz}\leq 1 \;\Leftrightarrow\; [2,1,0]\leq \frac{1}{2}[1,1,1]+\frac{1}{2}[3,0,0],$$

which is true by Schur $\square$